were also reduced by eliminating the field strength. These results advocate that the resonance of the beam can be avoided by employing the ER fluid as an actuator in the control system. It is finally noted that the derivation of phenomenological governing equations of the proposed smart structure using the data observed in this experimental investigation is yet to be further studied.

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Determining Free-Free Modes from Experimental Data of Constrained Structures

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Introduction

IT is very difficult to determine the frequencies and mode shapes of an unconstrained structure with experimental methods. Przemieniecki, in his monograph,[1] used vibration test data of a structure to determine the frequencies and mode shapes of the same structure in a free-free state. However, Przemieniecki's method requires that all of the frequencies and mode shapes of the constrained structure be obtained from the vibration tests, which is very difficult for a large structure. The lower frequencies and corresponding mode shapes can easily be obtained with much more accuracy from the vibration tests of the constrained structure, whereas the higher frequencies and the corresponding mode shapes are difficult to obtain. Some low-frequency modes can be obtained, whereas the high-frequency modes are truncated.

In this Note, a method for improving Przemieniecki's method is presented that extends Przemieniecki's method to deal with the problem of modal truncation. This method is of importance in extracting the natural frequencies and mode shapes of unconstrained structures, such as aircraft, rockets, and satellites, etc., from the experimental data of the corresponding constrained structures.

Przemieniecki's Method

Consider an unconstrained structure as shown in Fig. 1. Assume that the displacements on the unconstrained structure are partitioned into U_x and U_y. Furthermore, we assume that vibration tests to determine frequencies and mode shapes can be performed while supporting the structure in such a manner that $U_y = 0$ and that all rigid-body degrees of freedom are excluded. As an example we may use a rocket attached to its launch pad, shown in Fig. 1. The displacements U_y will be those associated with the attachment points, whereas U_x will represent all of the remaining displacements, the number of which will depend on the idealization of the structure.[1]

According to Ref. 1, we get

$$\left(\begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} - \omega^2 \begin{bmatrix} K_{xx}\bar{P}_e\bar{\Omega}_e^{-2}\bar{P}_e^{-1} & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix}\right)\begin{bmatrix} q_x \\ q_y \end{bmatrix} = 0 \qquad (1)$$

$$\bar{P}_e = [\bar{\Psi}_1, \bar{\Psi}_2, \ldots, \bar{\Psi}_m] \qquad (2)$$

$$\bar{\Omega}_e^2 = \mathrm{diag}[\bar{\omega}_1^2, \bar{\omega}_2^2, \ldots, \bar{\omega}_m^2] \qquad (3)$$

where $\bar{\Psi}_1, \bar{\Psi}_2, \ldots, \bar{\Psi}_m$ are the mode shapes for the constrained system with $U_y = 0$ and $\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_m$ are the associated frequencies.

Equation (1) can be used to determine the frequencies ω and modes $\{q_x^T \; q_y^T\}$ for the unconstrained structure. The stiffness matrices K_{xx}, K_{xy}, K_{yx}, and K_{yy} can be obtained from static tests on the constrained structure by first determining the influence coefficients for the directions of U_x with $U_y = 0$ and then deriving the required stiffness matrix. The modes $\bar{P}_e$ and frequencies $\bar{\Omega}_e$ are obtained from vibration tests, which allow us to determine the matrix product $K_{xx}\bar{P}_e\bar{\Omega}_e^{-2}\bar{P}_e^{-1}$; however, the submatrices M_{xy}, M_{yx}, and M_{yy} must be calculated, since no reliable direct experimental techniques are available to measure mass matrices. Alternatively, the structure can be supported in other locations and the vibration tests carried out to determine the remaining mass submatrices.[1]

Treatment of the Unavailable Modes

In the vibration tests, only the small number of lower frequencies and the associated mode shapes can be accurately and easily identified, whereas the higher frequencies and the associated mode shapes are difficult to identify accurately. Accordingly, the higher frequencies and the associated mode shapes are often unavailable. It can be seen that Przemieniecki's method cannot be implemented in the case of the higher frequencies and the corresponding mode shapes truncated, because the terms in Eq. (1) include the inverse of the modal matrix $\bar{P}_e^{-1}$, which requires all of the mode shapes. An approach to the modal truncation problem encountered in using Przemieniecki's method is presented in the following.

Let us focus on $K_{xx}\bar{P}_e\bar{\Omega}_e^{-2}\bar{P}_e^{-1}$. Set

$$W = K_{xx}\bar{P}_e\bar{\Omega}_e^{-2}\bar{P}_e^{-1} \qquad (4)$$

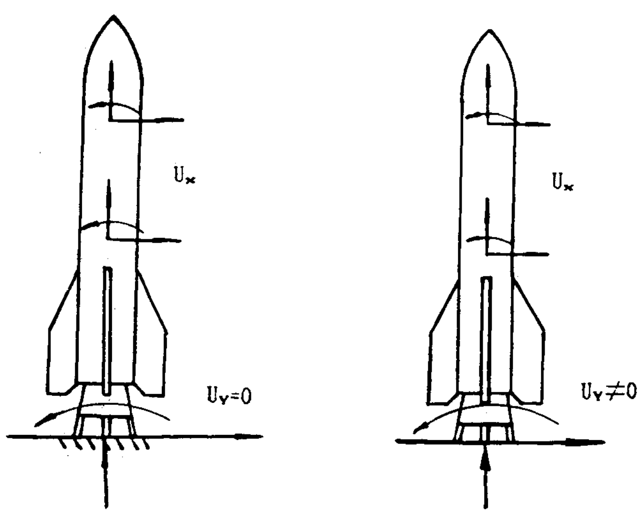

Fig. 1 Unconstrained structure.

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Recalling

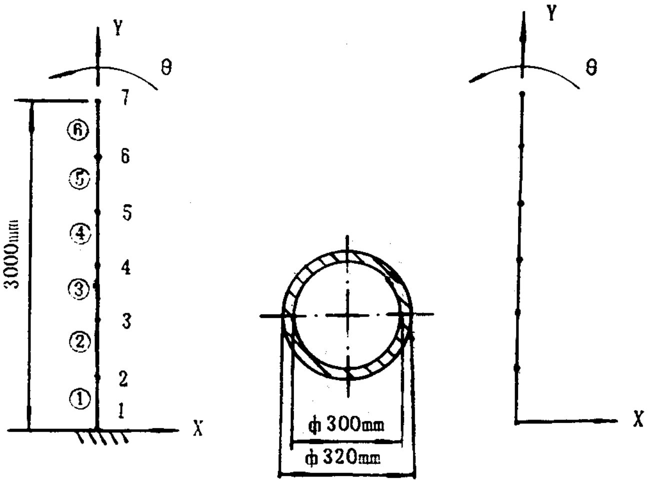
$$\bar{P}_e^T K_{xx} \bar{P}_e = \bar{\Omega}_e^{-2} \tag{5}$$

then we obtain from Eq. (5)

$$\bar{P}_e^{-1} = \bar{\Omega}_e^{-2} \bar{P}_e^T K_{xx} \tag{6}$$

Substituting Eq. (6) into Eq. (4) yields

$$W = K_{xx} \bar{P}_e \bar{\Omega}_e^{-4} \bar{P}_e^T K_{xx} \tag{7}$$



Young's modulus of elasticity $E = 6.9 \times 10^{10} N/m^2$
Mass density $\rho = 2714.3\ kg/m^3$

Fig. 2 Experimental data for the clamped-free beam.

Partition the matrices $\bar{P}_e$ and $\bar{\Omega}_e^{-4}$ in terms of the available part and the unavailable part; then Eq. (7) can be expressed as

$$\begin{aligned} W &= K_{xx}[\bar{P}_{el} \mid \bar{P}_{eh}]\ \mathrm{diag}(\bar{\Omega}_L^{-4}, \bar{\Omega}_h^{-4})(\bar{P}_{el} \mid \bar{P}_{eh})^T K_{xx} \\ &= K_{xx} \bar{P}_{el} \bar{\Omega}_l^{-4} \bar{P}_{el}^T K_{xx} + K_{xx} \bar{P}_{eh} \bar{\Omega}_h^{-4} \bar{P}_{eh}^T K_{xx} \end{aligned} \tag{8}$$

where $\bar{P}_{el}$ is the mode shape matrix with the first L available low-frequency modes as its columns, $\bar{\Omega}_L$ is the diagonal matrix with the first L available frequencies on its diagonals, $\bar{P}_{eh}$ is the mode shape matrix with the unavailable high-frequency modes as its columns, and $\bar{\Omega}_h$ is the diagonal matrix with the unavailable frequencies on its diagonals. Since

$$\bar{\Omega}_h^{-4} \ll \bar{\Omega}_L^{-4} \tag{9}$$

Equation (8) can be approximated as

$$W \approx K_{xx} \bar{P}_{el} \bar{\Omega}_l^{-4} \bar{P}_{el}^T K_{xx} \equiv W_a \tag{10}$$

Substituting Eq. (10) into the eigenproblem Eq. (1), we obtain

$$\begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} - \omega^2 \begin{bmatrix} W_a & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix} \begin{bmatrix} q_x \\ q_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

It can be seen that Eq. (11) does not contain the unavailable higher frequencies and mode shapes and does not include the inverse of $\bar{P}_e$. Accordingly, Eq. (11) can be used without involving the higher unavailable modes to extract the eigensolutions associated with the free-free structures.

Numerical Example

As an indication of the accuracy of the method presented in this paper, a numerical example is used to illustrate the validity of the method. In this example, a free-free beam vibration in plane is of concern. The first five nonrigid frequencies and

Table 1 Comparison of eigensolutions

		Exact solutions		
		Eigenvalues		
0.14583370E+04	0.11113289E+05	0.43054846E+05	0.11926817E+06	0.26556890E+06
		Eigenvectors		
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
−0.91736241E−01	−0.91993467E−01	0.92650632E−01	0.93440410E−01	−0.88750762E−01
0.23684865E−01	0.40178007E−01	−0.56713761E−01	−0.74017843E−01	0.86853256E−01
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
−0.21929839E−01	0.20207115E−01	−0.49473485E−01	−0.62278888E−01	0.53985338E−01
0.22096566E−01	0.30034077E−01	−0.24464672E−01	−0.28716896E−02	−0.29402366E−01
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.33996974E−01	0.59751230E−01	−0.15682605E−01	0.47974146E−01	−0.65386340E−01
0.13942124E−01	−0.57838283E−02	0.37968992E−01	0.36047203E−01	0.16837579E−01
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.55759538E−01	−0.32460416E−15	0.66018429E−01	0.61405608E−16	0.68126138E−01
−0.20754844E−15	−0.27600654E−01	−0.25486309E−16	−0.51857775E−01	0.24538602E−16
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.33996974E−01	−0.59751230E−01	−0.15682605E−01	−0.47974146E−01	−0.65386340E−01
−0.13942124E−01	−0.57838283E−02	−0.37968992E−01	0.36047203E−01	−0.16837579E−01
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
−0.21929839E−01	−0.20207115E−01	−0.49473485E−01	0.62278888E−01	0.53985338E−01
−0.22096566E−01	0.30034077E−01	0.24464672E−01	−0.28716896E−02	0.29402366E−01
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
−0.91736241E−04	0.91993467E−01	0.92650632E−01	−0.93440410E−01	−0.88750762E−01
−0.23684865E−01	0.40178007E−01	0.56713761E−01	−0.74017843E−01	−0.86853256E−01
		Eigenvalues obtained by the present method		
Case 1: (the first 12 modes retained)				
0.14583906E+04	0.11113555E+05	0.43055359E+05	0.11926800E+06	0.26556511E+06
Case 2: (the first 9 modes retained)				
0.14595810E+04	0.11119184E+05	0.43068028E+05	0.11928571E+06	0.26557672E+06
Case 3: (the first 6 modes retained)				
0.14797071E+04	0.11199063E+05	0.43207465E+05	0.11940393E+06	0.18584581E+07

Table 2 **Comparison of errors for eigensolutions**

Case 1: (the first 12 modes retained)				
Errors of eigenvalues, %				
0.36754193E-02	0.23935308E-02	0.11915035E-02	0.14253593E-03	0.14271249E-02
Errors of eigenvectors				
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.51380000E-05	0.68220000E-05	0.75880000E-05	0.75580000E-05	0.65840000E-05
0.57300000E-06	0.13950000E-05	0.26090000E-05	0.38520000E-05	0.45700000E-05
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.35410000E-05	0.35730000E-05	0.29580000E-05	0.28520000E-05	0.26900000E-05
0.41400000E-06	0.20500000E-06	0.13710000E-05	0.44599000E-05	0.70000000E-05
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.16490000E-05	0.61000000E-07	0.10090000E-05	0.39700000E-06	0.56800000E-06
0.30200000E-06	0.11128000E-05	0.58090000E-05	0.13164000E-04	0.17980000E-04
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.31000000E-07	0.13812469E-05	0.36000000E-06	0.53876547E-06	0.38500000E-06
0.69023433E-07	0.23010000E-05	0.72960426E-05	0.13688000E-04	0.17529410E-04
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.79000000E-06	0.38500000E-06	0.27000000E-07	0.14510000E-05	0.29910000E-05
0.17100000E-06	0.18547000E-05	0.41530000E-05	0.84970000E-05	0.12501000E-04
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.53800000E-06	0.18000000E-06	0.99700000E-06	0.21350000E-05	0.17950000E-05
0.21400000E-06	0.64000000E-07	0.90000000E-07	0.88290000E-06	0.26400000E-06
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.12400000E-06	0.68500000E-06	0.20810000E-05	0.36140000E-05	0.58290000E-05
0.43000000E-07	0.19940000E-05	0.55190000E-05	0.10777000E-04	0.15517000E-04
Case 2: (the first 9 modes retained)				
Errors of eigenvalues, %				
0.85302643E-01	0.53044603E-01	0.30616763E-01	0.14706355E-01	0.29446219E-02
Errors of eigenvectors				
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.12404800E-03	0.16200700E-03	0.18079600E-03	0.18598000E-03	0.15041200E-03
0.13933000E-04	0.34338000E-04	0.67087000E-04	0.10770000E-03	0.12221100E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.85178000E-04	0.80767000E-04	0.54995000E-04	0.34289000E-04	0.25308000E-04
0.10137000E-04	0.66210000E-05	0.26207000E-04	0.94455500E-04	0.14377900E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.39331000E-04	0.10093000E-04	0.51537000E-04	0.58237000E-04	0.50196000E-04
0.13703000E-04	0.13368900E-04	0.17590000E-04	0.60764000E-04	0.73010000E-04
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.86700000E-06	0.32008017E-04	0.39920000E-05	0.14522373E-04	0.11800000E-05
0.12812251E-04	0.12499000E-04	0.28354452E-04	0.87927000E-04	0.11429769E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.16515000E-04	0.97000000E-07	0.33367000E-04	0.24475000E-04	0.31348000E-04
0.16230000E-05	0.11592400E-04	0.12720000E-05	0.14940000E-05	0.85930000E-05
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.12761000E-04	0.93700000E-05	0.10458000E-04	0.16181000E-04	0.11131000E-04
0.61220000E-05	0.10338000E-04	0.26381000E-04	0.66465900E-04	0.73053000E-04
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.00000000E+00
0.23770000E-05	0.63440000E-05	0.26466000E-04	0.40114000E-04	0.60188000E-04
0.33360000E-05	0.18798000E-04	0.39633000E-04	0.80739000E-04	0.10312300E-03

the associated mode shapes for the free-free beam can be determined using the experimental data for the clamped-free beam, as shown in Fig. 2.

In this example, the first L natural frequencies and eigenvectors obtained by computation are taken as the experimental data for the clamped-free beam. For the sake of comparison, the exact frequencies and the associated mode shapes of the free-free beam are also calculated. In the whole process this beam is discretized into six beam elements.

The exact nonrigid-body eigensolutions associated with the free-free structures are listed in Table 1. The nonrigid-body eigensolutions obtained using the present method, Eqs. (10) and (11), are also summarized in Table 1, in which the three cases, i.e., $L = 6$, 9, and 12, respectively, are considered. The errors of eigensolutions obtained for the three cases using the present method are given in Table 2.

It can be observed from Tables 1 and 2 that the present method is valid. The errors of eigensolutions for the free-free beam due to the unavailable higher frequencies and the mode shapes associated with the clamped-free beam vary with the L that is the number of the eigensolutions retained in $\bar{P}_{e1}$ and Ω_{e1} of Eqs. (10) and (11) for the clamped-free beam. The smaller the L, the larger the errors. For example, when $L = 6$ and 9, the errors of the first eigenvalue are 1.465 and 0.085%, respectively, whereas when $L = 12$, the error is reduced to 0.0036%. As can be seen from this example, if one hopes to obtain accurately the first J nonrigid eigensolutions for the free-free structure, the number of the eigensolutions for the same constrained structure, retained in Eqs. (10) and (11), should not be less than $J + 2$. For example, in this numerical example (case 3), when $L = 6$, the fifth eigenvalue and its eigenvector obtained by the present method are inaccurate; however, the first four eigenvalues and their eigenvectors are correct.

Concluding Remarks

Przemieniecki proposed a method[1] for determining the natural vibration characteristics of free-free structures using the vibration test data for the same structure supported rigidly on the ground. However, it is impossible to obtain all of the frequencies and mode shapes in vibration tests. This paper extends Przemieniecki's method to deal with the modal truncation problem. A numerical example is given that shows good agreement of the present method with the exact solutions in the range of low-frequency modes of interest.

Acknowledgments

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